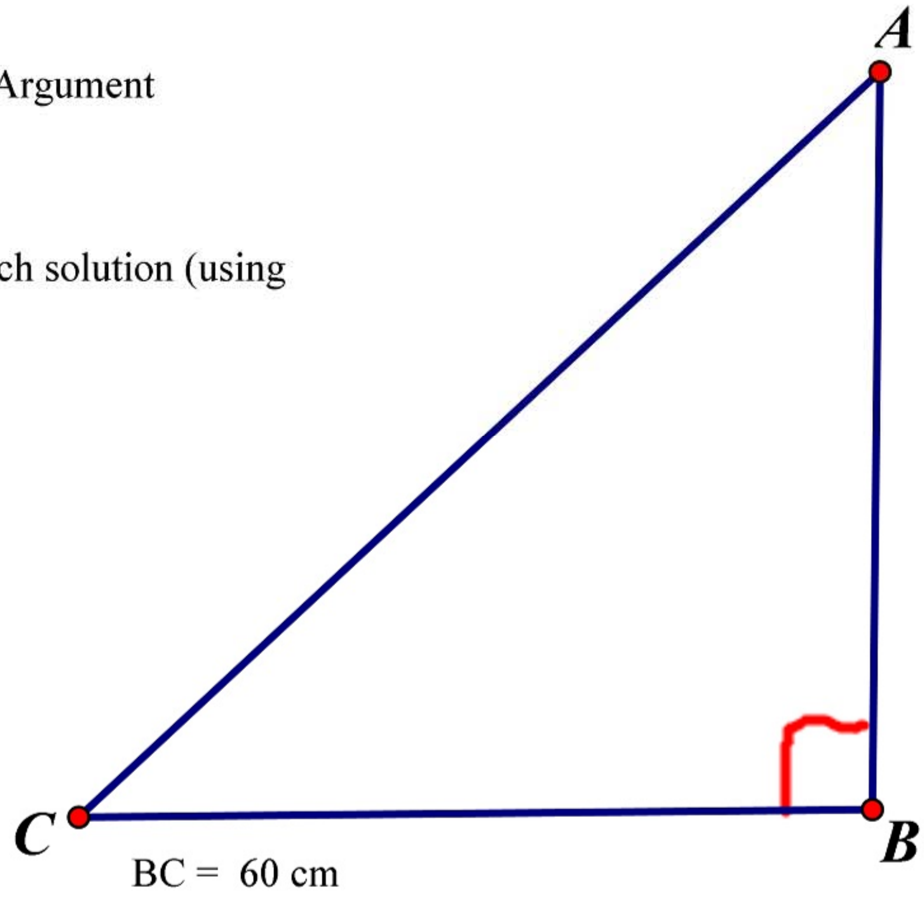
Solved by two students
1. Mueez Khan (discrete Math, MTH 213, Fall 2017)
2. Yousuf Abo Rahama (Abstract Algebra, MTH 320, Fall 2017)

ABC is a right triangle. Length of CB = 60.
Let Z be the length of AC and Y be the length of AB.

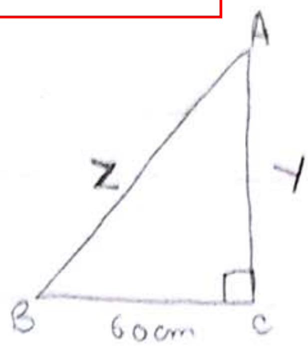
Find all possible PAIRS (Z, Y). where Z, Y are POSITIVE INTEGERS.

To get the 100AED AWARD you need to give me a correct Mathematical Argument showing how one can obtain all possible pairs (Z, Y).

Try and error, Calculator, Computer Program ARE NOT ACCEPTED..
For Example if (5, 2) is a solution, you need to clarify how did you get such solution (using Mathematical Argument) .



Solution one by Mueez Khan (Discrete Math, MTH 213, Fall 2017)



2×1800

6×600 — 4×900

8×450

10×360

12×300

18×200

20×180

24×150

30×120

36×100

40×90

50×72

60×60

Pairs of even factors of 3600

$$c^2 = a^2 + b^2$$
$$c^2 - a^2 = b^2$$
$$(c-a)(c+a) = b^2$$
$$(c-a)(c+a) = 3600$$

$b = BC = 60$

m and n must be pairs of even factors of 3600.

Suppose

$c^2 - a^2 = b^2 \Rightarrow m^2 - n^2 = b^2$

$m^2 - n^2 = 3600$

$(m-n)(m+n) = 3600$

Since both brackets are even, they must be divisible by 2.

Let. $c = \left(\frac{m+n}{2}\right)$ $a = \left(\frac{m-n}{2}\right)$

even. m and n must be pairs of even factors of 3600.

1) 1800×2

$c = \frac{m+n}{2} = 901$ $a = \frac{m-n}{2} = 899$

Pair = (899, 901)

2) 600×6

$a = \frac{600-6}{2}$ $c = \frac{600+6}{2}$

$a = 297$ $c = 303$

(297, 303)

Pairs are not in order, bigger number is Z, smaller number is Y. But OK you are consistent... No big deal!

3) 8×450

$a = \frac{450-8}{2}$ $\quad c = \frac{450+8}{2}$

$= 221$ $\quad = 229$

$(221, 229)$

4) 10×360

$a = \frac{360-10}{2}$ $\quad c = \frac{360+10}{2}$

$= 175$ $\quad = 185$

$(175, 185)$

5) 12×300

$a = \frac{300-12}{2}$ $\quad c = \frac{300+12}{2}$

$= 144$ $\quad = 156$

$(144, 156)$

6) 18×200

$a = \frac{200-18}{2}$ $\quad c = \frac{200+18}{2}$

$= 91$ $\quad 109$

$(91, 109)$

7) 20×180

$a = \frac{180-20}{2}$ $\quad c = \frac{180+20}{2}$

$= 80$ $\quad = 100$

$(80, 100)$

8) 150×24

$a = \frac{150-24}{2}$ $\quad b = \frac{150+24}{2}$

$= 63$ $\quad = 87$

$(63, 87)$

9) 30×120

$a = \frac{120-30}{2}$ $\quad c = \frac{120+30}{2}$

$= 45$ $\quad = 75$

$(45, 75)$

10) 36×100

$a = \frac{100-36}{2}$ $\quad c = \frac{100+36}{2}$

$= 32$ $\quad = 68$

$(32, 68)$

11) 40×90

$a = \frac{90-40}{2}$ $\quad c = \frac{90+40}{2}$

$= 25$ $\quad = 65$

$(25, 65)$

12) 50×72

$a = \frac{72-50}{2}$ $\quad c = \frac{72+50}{2}$

$= 11$ $\quad 61$

$(11, 61)$

13) 4×900 $\quad a = \frac{900-4}{2}$ $\quad c = \frac{900-4}{2}$

$= 448$ $\quad 452$ $\quad (448, 452)$

Solution II by Yousuf Abo Rahama (Abstract Algebra MTH 320, Fall 2017)

Taking into consideration the degenerate case $Z = 60, Y = 0$ the problem can be reformulated as finding the solution set of

$$-Y^2 + Z^2 = 60^2 \;, \quad Z, Y \in \mathbb{N}$$

Simplifying the Left Hand side we get

$(Z - Y)(Z + Y) = 60^2 \Rightarrow Z - Y, Z + Y$ are two factors of 60^2 such that their product is 60^2.

Now, let $K_1 = Z - Y \ldots (1)$, $K_2 = Z + Y \ldots (2) \Rightarrow K_2 = \dfrac{60^2}{K_1}$, $K_1 \mid 60^2$

and K_1, K_2 are both even since they are the addition and the difference of the same two numbers and their product is even.

Now from (1) and (2) we get $Z = K_1 + Y \Rightarrow K_2 = K_1 + 2Y$

Thus, $\dfrac{60^2}{K_1} = K_1 + 2Y \Rightarrow Y = \dfrac{60^2 - K_1^2}{2K_1} \Rightarrow Z = \dfrac{60^2 + K_1^2}{2K_1}$

Since K_1 is even we will write it as $K_1 = 2n$ where $n \mid \dfrac{60^2}{2}$ and to have the pairs non negative we set $n \leq 30$ and hence the solution set is

$$S = \left\{ \left(\frac{900 + n^2}{n}, \frac{900 - n^2}{n} \right) \Big| \, 0 < n \leq 30, \text{ and } n \mid 900 \right\}$$

Note $1 \le n < 30$

so n = 1,2,3,4,5,6,9,10,12,15,18,20,25. To be more precise, we have exactly 13 pairs (901,899), (452,448), (303,297), (229,221), (185,175), (156,144), (109,91), (100,80), (87,63), (75,45), (68,32), (65,25), (61,11)